International Journal on Recent and Innovation Trends in Computing and Communication
ISSN: 2321-8169 Volume: 10 Issue:05
Article Received: 25 February 2022 Revised: 12 March 2022 Accepted: 30 April 2022

Properties of Weakly Uniformly Recurrence on Hyperspaces

Amalraj.P[1*], Vinod Kumar P.B[2]

[1*]Assistant Professor, Mathematics, S. D. College, Alappuzha, Kerala, India, APJ Abdul Kalam Technological University, Kerala, India
[2]Professor, Department of Mathematics, Rajagiri School of Engineering And Technology, Kerala, India, APJ Abdul Kalam Technological University, Kerala, India

Abstract: Let X be an infinite compact metric space without isolated points and $f: X \to X$ continuous on X . Consider the space $K(X)$, the set of all compact subsets of X with Hausdorff metric H and the induced map $\tilde{f}:K(X) \to K(X)$ defined by $\tilde{f}(A) = f(A)$. In this paper we establish some properties of weakly uniformly recurrence and asymptotically sensitive dependence on initial conditions (to be defined below) on hyper space $K(X)$.

1. Introduction:

All spaces considered in this paper are compact metric space without isolated points. Consider the space $K(X)$, the set of all compact subsets of X with Hausdorff metric H and the induced map $\tilde{f}:K(X) \to K(X)$ defined by $\tilde{f}(A) = f(A)$.The Hausdorff metric H is defined by $H(A, B) = \inf \sup\{d(a, b), a \in A, b \in B\}$(see.1). It is known that the continuity of f implies the continuity of $\tilde{f}$ (see.1). We refer $\tilde{f}$ as the induced map of f.

2.Asymptotically sensitive Dependence On Initial Conditions:

Theorem 2.1

Let $\tilde{f}$ is transitive then there exist a Cantor set C such that for any $A \in K(X)$ and any positive integer M, $\lim_{N\to\infty} H(\tilde{f}^{M+N}(A), \tilde{f}^{N}(C)) \geq H(A, \tilde{f}^{M}(A))$.

Proof:

Let $A \in K(X)$ and m is any positive integer. Let $\{a_n\}$ be an arbitrary decreasing sequence of positive numbers with $a_n \to 0$. Let $\tilde{f}:K(X) \to K(X)$ be transitive, then there exist a cantor set $C \subseteq X$ with $orb(\tilde{f}, C)$ is dense in $K(X)$(see 2). So there exist for every a_n, a positive integer b_n such that $\tilde{f}^{b_n}(C)$ so close to A such that $H\left(\tilde{f}^{b_n}(C), A\right) < \frac{a_n}{2}$ and $H(\tilde{f}\left(\tilde{f}^{b_n}(C), \tilde{f}^{M}(A)\right) < \frac{a_n}{2}$.

This implies that $H\left(\tilde{f}^{b_n}(C), \tilde{f}^{b_n} + M(C)\right) \geq H\left(A, \tilde{f}^{M}(A)\right) - a_n$ (By Triangle Inequality).

Hence the result.

Definition 2.1

Let δ be a positive number and A be a subset of X with at least two points and $f: X \to X$ We say that f is asymptotically sensitive dependence on initial conditions, if for every $x \in X$ and every open neighbourhood $N(x)$ of x, there is a point $y \in N(x)$ with $\limsup_{n\to\infty} d(f^n(x), f^n(y)) \geq \delta$.

Definition 2.2

Let $f: X \to X$. f has δ- sensitive dependence on initial conditions, if for every point $x \in X$ and every positive number ϵ there is a point $y \in X$ with $d(x, y) < \epsilon$ and a positive integer
n such that $d\big(f^n(x), f^n(y)\big) \geq \delta$.

Theorem 2.2

Assume that $\tilde{f}$ is transitive. Then $\tilde{f}$ has sensitive dependence on initial conditions if and only if $\tilde{f}$ has asymptotically sensitive dependence on initial conditions.

Proof:

If $\tilde{f}$ has asymptotically sensitive dependence on initial conditions, then it is clear that $\tilde{f}$ also has sensitive dependence on initial conditions.
So assume that $\tilde{f}$ has sensitive dependence on initial conditions. Since $\tilde{f}$ is transitive, there exist a Cantor set C in $K(X)$ with dense orbit. Then for some positive integer δ, $\tilde{f}$ has δ- sensitive dependence on initial conditions.
Let $A \in K(X)$ be any point and let $N(A)$ be any neighbourhood of A. Then there exist a point $B \in N(A)$ and a positive integer s such that $H\left(\tilde{f}^s(A), \tilde{f}^s(B)\right) > \delta$. Since $\tilde{f}$ is continuous and since orbit of C is dense in $K(X)$, there exist $U \in orb\big(\tilde{f}, C\big) \cap N(A)$ which is so close to A with $H\left(\tilde{f}^s(U), \tilde{f}^s(A)\right) < \frac{1}{2}\left[H\left(\tilde{f}^s(A), \tilde{f}^s(B)\right) - \delta\right]$. But since the orbit of U is dense in $K(X)$, there exist a positive integer r such that $\tilde{f}^r(U)$ is close to the point B with
$\tilde{f}^r(U) \in N(A)$ and $H\left(\tilde{f}^{s+r}(U), \tilde{f}^s(B)\right) < \frac{1}{2}\left[H\left(\tilde{f}^s(A), \tilde{f}^s(B)\right) - \delta\right]$.

This implies that $H\left(\tilde{f}^{s+r}(U), \tilde{f}^{s}(B)\right) > \delta$ (By Triangle Inequality).
Thus, we have either $\limsup_{n\to\infty} H(\tilde{f}^n(A), \tilde{f}^n(U)) > \frac{\delta}{2}$ or $\limsup_{n\to\infty} H(\tilde{f}^n(A), \tilde{f}^n(\tilde{f}^r(U)) > \frac{\delta}{2}$.
Since $U \in N(A)$ and $\tilde{f}^r(U) \in N(A)$, we have the result.

3.Weakly Uniformly Recurrence On Hyperspaces
Definition 3.1
Let $f: X \to X$. We say that f is weakly uniformly recurrent with respect to the metric d, for $x \in X$, there exist a strictly increasing sequence $\{n_i\}$ of natural numbers such that $d\left(x, f^{n_i}(x)\right) \to 0$.

Theorem 3.1
Assume that $\tilde{f}$ is weakly uniformly recurrent with respect to the Hausdorff metric H. Then for any two distinct compact subsets A and B in X,
$\limsup_{m\to\infty} H\left(\tilde{f}^m(A), \tilde{f}^m(B)\right) \geq H(A,B) > 0$.
In particular, $\tilde{f}$ is one-to-one.

Proof:
Assume that $\tilde{f}$ is weakly uniformly recurrent with respect to the metric H. Assume on the contrary that there exist two distinct compact sets A and B in $K(X)$ with
$\limsup_{m\to\infty} H\left(\tilde{f}^m(A), \tilde{f}^m(B)\right) < H(A,B)$. Then there exist a positive number $\epsilon < 1$ such that
$\limsup_{m\to\infty} H\left(\tilde{f}^m(A), \tilde{f}^m(B)\right) < \epsilon H(A,B)$. So there is a positive integer $N(\epsilon)$ such that
$H\left(\tilde{f}^n(A), \tilde{f}^n(B)\right) < \epsilon H(A,B), \forall\, n \geq N(\epsilon)$. Since $\tilde{f}$ is weakly uniformly recurrent , there exist a positive integer $p > N(\epsilon)$ such that $H\left(A, \tilde{f}^p(A)\right) < 4^{-1}(1-\epsilon)H(A,B)$ and $H\left(B, \tilde{f}^p(B)\right) < 4^{-1}(1-\epsilon)H(A,B)$.
For this p, we have $H(A,B) \leq H\left(A, \tilde{f}^p(A)\right) + H\left(\tilde{f}^p(A), \tilde{f}^p(B)\right) + H\left(B, \tilde{f}^p(B)\right) \leq 4^{-1}(1-\epsilon)H(A,B) + \epsilon H(A,B) + 4^{-1}(1-\epsilon)H(A,B) = (1+\epsilon)2^{-1}H(A,B) < H(A,B)$
is a contradiction. There fore we conclude that
$\limsup_{m\to\infty} H\left(\tilde{f}^m(A), \tilde{f}^m(B)\right) \geq H(A,B) > 0$.

Theorem 3.2
Assume that $\tilde{f}$ is weakly uniformly recurrent with respect to the metric H and assume that $\limsup_{m\to\infty} H\left(\tilde{f}^m(A), \tilde{f}^m(B)\right)$ is finite for every A and B in $K(X)$. Then $H^*: K(X) \times K(X) \to \mathbb{R}$ defined by $H^*(A,B) = \limsup_{m\to\infty} H\left(\tilde{f}^m(A), \tilde{f}^m(B)\right)$ is a metric on $K(X)$.

Proof:
Clear from the definition of H^*.

Theorem 3.3
Assume that $\tilde{f}$ is weakly uniformly recurrent with respect to the metric H.Then the following hold.
(i) $H^*(A,B) \geq H(A,B)$
(ii) $\tilde{f}$ is an isometry with respect to the metric H^*
(iii) $\tilde{f}$ is weakly uniformly recurrent with respect to the metric H^*.

Proof:
(i) Clear from **Theorem 3.1**
(ii) $H^*\left(\tilde{f}(A), \tilde{f}(B)\right) =$
$\limsup_{m\to\infty} H\left(\tilde{f}^m(\tilde{f}(A)), \tilde{f}^m\left(\tilde{f}(B)\right)\right) =$
$\limsup_{m\to\infty} H\left(\tilde{f}^{m+1}(A), \tilde{f}^{m+1}(B)\right) = H(A,B)$.
ie, H^* is an isometry.
(iii) Since $\tilde{f}$ is weakly uniformly recurrent with respect to the metric H, we have for $A \in K(X)$, there is a strictly increasing sequence of natural numbers $\{n_i\}$such that $H\left(A, \tilde{f}^{n_i}(A)\right) \to 0$. That is, for a given $\delta > 0$, there exist a natural number $p > 0$ such that $H\left(A, \tilde{f}^s(A)\right) < \delta, \forall s \geq p$.
Then it is clear that $H\left(\tilde{f}^m(A), \tilde{f}^{s+m}(A)\right) < \delta$, $\forall s \geq p, m > 0$.
Consequently $\limsup_{m\to\infty} H\left(\tilde{f}^m(A), \tilde{f}^m(B)\right) \leq \delta$.
Hence the result.

References:

1) A. Illaness, S.B.Nadler Jr, Hyperspaces:Fundamntals and Recent Advances,,Pure App.math-Vol 216(1999)
2) Compcat Sets with Dense Orbit in 2^X, Palona Hernandez, Jefferson King,,Hector Mendez, Topology Proceedings, vol.40(2012)