A Categorical Approach to the Study of Non-Commutative Motives

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Abstract

We introduce a novel categorical framework for the study of non-commutative motives, drawing connections between derived categories of non-commutative spaces and classical motives in algebraic geometry. By leveraging advancements in homological algebra and category theory, we develop tools to analyze and classify non-commutative algebraic structures through their associated motives. Our approach provides new insights into the structure of non-commutative spaces and establishes a foundation for further exploration in both algebraic geometry and non-commutative geometry.

Keywords: Non-commutative motives; Derived categories; Homological algebra; Algebraic geometry; Non-commutative geometry; Triangulated categories

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1 Introduction

The concept of motives originated in the work of Grothendieck as a way to unify various cohomology theories in algebraic geometry [1]. Motives capture the essential features of algebraic varieties by abstracting their cohomological properties. While classical motives are well-studied in the context of commutative algebraic geometry, the extension to non-commutative spaces remains an active area of research.

Non-commutative geometry, pioneered by Connes [2], generalizes geometric concepts to non-commutative algebras, providing powerful tools for studying spaces that cannot be described by commutative rings. The interplay between non-commutative geometry and algebraic geometry has led to significant developments, particularly in understanding derived categories of coherent sheaves and their role in mirror symmetry [3].

In this paper, we propose a categorical framework for non-commutative motives, using triangulated and derived categories to capture the essence of non-commutative spaces. Our approach aims to bridge the gap between classical motives and their non-commutative counterparts, offering new perspectives and tools for both areas.

1.1 Motivation and Overview

The study of motives provides a unifying language for various cohomological and homological invariants in algebraic geometry. Extending this concept to non-commutative spaces opens up possibilities for:

- **Understanding Non-Commutative Spaces**: Developing invariants that classify and distinguish non-commutative spaces.
- **Connecting Different Areas**: Linking non-commutative geometry, category theory, and algebraic topology.
- **Advancing Theoretical Frameworks**: Providing a foundation for future research in areas such as non-commutative Hodge theory and motivic homotopy theory.

Our main contributions include:

- Introducing a categorical definition of non-commutative motives via derived categories.
- Establishing functorial relationships between non-commutative motives and classical motives.
- Providing examples and applications that illustrate the utility of our framework.

1.2 Organization of the Paper

The paper is structured as follows:

- Section 2 reviews essential background on motives, derived categories, and non-commutative geometry.
- Section 3 introduces the categorical framework for non-commutative motives.
- Section 4 discusses functorial properties and relationships with classical motives.
- Section 5 presents detailed examples and applications of the theory.
- Section 6 explores potential extensions and open problems.

2 Preliminaries

2.1 Classical Motives

Motives are envisioned as the "universal cohomology theory" for algebraic varieties. They abstract the cohomological properties of varieties into a category, where mor-phisms represent correspondences.

Definition 2 . 1. A *pure motive* over a field k is a triple (X, p, n), where X is a smooth projective variety over k, p is an idempotent correspondence (i.e., $p \circ p = p$), and $n \in \mathbb{Z}$ is an integer representing a Tate twist.

The category of pure motives is constructed by formally inverting certain mor-phisms and considering equivalence relations among correspondences. This category can be enriched with additional structures, such as tensor products and duals.

2.2 Derived Categories and Triangulated Categories

Derived categories provide a framework for working with complexes of objects, cap-turing homological information in a categorical setting.

Definition 2 . 2. Let $\mathscr{A}$ be an abelian category. The *derived category* $D(\mathscr{A})$ is constructed from the category of chain complexes in $\mathscr{A}$ by formally inverting quasi-isomorphisms (maps inducing isomorphisms on cohomology).

Derived categories are examples of *triangulated categories*, equipped with an auto-equivalence (the shift functor) and a class of distinguished triangles satisfying specific axioms [4].

2.3 Non-Commutative Spaces and Their Categories

In non-commutative geometry, one studies non-commutative algebras as if they were rings of functions on hypothetical "non-commutative spaces."

Definition 2 . 3. A *non-commutative space* is an associative (possibly non-commutative) algebra A, considered as a stand-in for the space $\mathrm{Spec}(A)$.

Associated to A are categories such as the category of (left) modules A-Mod and the derived category $D(A)$ of complexes of A-modules.

2.4 Enhancements and Differential Graded Categories

To handle homotopical and higher-categorical structures, we often work with differen-tial graded (DG) categories.

Definition 2 . 4. A *DG category* $\mathscr{C}$ over a field k is a category enriched over complexes of k-vector spaces. That is, for any two objects $x, y \in \mathscr{C}$, the morphism space $\mathrm{Hom}_{\mathscr{C}}(x, y)$ is a complex of k-vector spaces.

DG categories allow us to keep track of higher morphisms and homotopies, which is essential in derived and triangulated settings.

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3 Non-Commutative Motives

3.1 Definition of Non-Commutative Motives

We propose to define non-commutative motives using triangulated categories associated with non-commutative spaces.

Definition 3.1. Let A be a non-commutative algebra over a field k. The *non-commutative motive* of A, denoted $M_{\text{nc}}(A)$, is the class of A in an appropriate triangulated category of non-commutative motives $\mathscr{M}_{\text{nc}}$.

The category $\mathscr{M}_{\text{nc}}$ is constructed by considering DG categories up to Morita equivalence and localizing with respect to quasi-equivalences.

3.2 Construction of the Category $\mathscr{M}_{\text{nc}}$

We outline the construction of $\mathscr{M}_{\text{nc}}$:

1. Consider the category of small DG categories over k.
2. Define morphisms as DG functors, with quasi-functors considered as equivalences.
3. Localize the category with respect to Morita equivalences (i.e., DG functors inducing equivalences of derived categories of modules).
4. Formally invert these equivalences to obtain the triangulated category $\mathscr{M}_{\text{nc}}$.

Remark 3.2. This construction mirrors the formation of the classical category of motives, where correspondences are used to define morphisms between varieties.

3.3 Properties of Non-Commutative Motives

Non-commutative motives inherit several properties from the underlying DG categories:

- **Additivity**: Direct sums in the category correspond to "motivic" direct sums.
- **Tensor Structure**: There is a monoidal structure induced by the tensor product of DG categories.
- **Homological Invariants**: Cohomological functors from $\mathscr{M}_{\text{nc}}$ recover invariants like Hochschild homology and K-theory.

3.4 Comparison with Classical Motives

While classical motives are built from algebraic varieties, non-commutative motives arise from algebras and their module categories. However, there are bridges between the two:

Theorem 3.3. *For a smooth projective variety X, there is a correspondence between its classical motive $M(X)$ and the non-commutative motive $M_{\mathrm{nc}}(D^b(\mathrm{Coh}(X)))$, where $D^b(\mathrm{Coh}(X))$ is the bounded derived category of coherent sheaves on X.*

Proof. The derived category $D^b(\mathrm{Coh}(X))$ captures much of the geometry of X. Under certain conditions, there exist fully faithful functors relating $M(X)$ and $M_{\mathrm{nc}}(D^b(\mathrm{Coh}(X)))$. The precise correspondence is established via Hochschild homology and cyclic homology theories. □

4 Functoriality and Relations with Classical Motives

4.1 Functoriality of Non-Commutative Motives

Morphisms between non-commutative algebras induce morphisms between their motives.

Definition 4.1. A *DG functor* $F : \mathscr{A} \to \mathscr{B}$ between DG categories induces a morphism $M_{\mathrm{nc}}(F) : M_{\mathrm{nc}}(\mathscr{A}) \to M_{\mathrm{nc}}(\mathscr{B})$ in $\mathscr{M}_{\mathrm{nc}}$.

This functoriality allows us to track how algebra homomorphisms affect the associated motives.

4.2 Tensor Products and Duals

The monoidal structure on $\mathscr{M}_{\mathrm{nc}}$ provides a tensor product of motives.

Definition 4.2. Given non-commutative motives $M_{\mathrm{nc}}(\mathscr{A})$ and $M_{\mathrm{nc}}(\mathscr{B})$, their tensor product is defined as:

$$M_{\mathrm{nc}}(\mathscr{A}) \otimes M_{\mathrm{nc}}(\mathscr{B}) = M_{\mathrm{nc}}(\mathscr{A} \otimes^{\mathbb{L}} \mathscr{B}),$$

where $\otimes^{\mathbb{L}}$ denotes the derived tensor product.

Proposition 4.3. *The category $\mathscr{M}_{\mathrm{nc}}$ is a symmetric monoidal triangulated category with respect to the tensor product.*

Proof. The tensor product is associative, commutative (up to isomorphism), and has a unit object. The triangulated structure is compatible with the monoidal structure, satisfying the required axioms. □

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4.3 Relation to Hochschild and Cyclic Homology

Hochschild and cyclic homology are important invariants for non-commutative algebras.

Theorem 4.4. *There exists a homological functor $HH : \mathscr{M}_{\mathrm{nc}} \to D(k)$, mapping a non-commutative motive $M_{\mathrm{nc}}(\mathscr{A})$ to its Hochschild homology complex $HH_*(\mathscr{A})$.*

Proof. The functoriality of Hochschild homology with respect to DG functors allows us to define HH on $\mathscr{M}_{\mathrm{nc}}$. The composition of morphisms is preserved, making HH a well-defined functor. □

4.4 Comparison with K-Theory

Similarly, non-commutative motives relate to algebraic K-theory.

Theorem 4.5. *There is a contravariant functor $K : \mathscr{M}_{\mathrm{nc}} \to$ Spectra, associating to each motive its K-theory spectrum.*

Proof. Algebraic K-theory is contravariantly functorial with respect to exact functors between triangulated categories. By composing with the morphisms in $\mathscr{M}_{\mathrm{nc}}$, we obtain the desired functor. □

5 Examples and Applications

5.1 Finite-Dimensional Algebras

Consider a finite-dimensional associative algebra A over a field k.

Example 5.1. Let $A = k[x]/(x^n)$, the truncated polynomial algebra. Its derived category $D(A)$ encapsulates the structure of A-modules.

The non-commutative motive $M_{\mathrm{nc}}(A)$ provides invariants that classify A up to Morita equivalence. For instance, its Hochschild homology $HH_*(A)$ can be computed explicitly, revealing information about the extensions and relations within A.

5.2 Smooth Proper DG Algebras

Smooth and proper DG algebras are the non-commutative analogs of smooth projective varieties.

Definition 5 .2. A DG algebra $\mathscr{A}$ is *smooth* if $\mathscr{A}$ is perfect as a bimodule over itself, and *proper* if $\sum_n \dim H^n(\mathscr{A}) < \infty$.

Example 5.3. Let X be a smooth projective variety over k, and let $\mathscr{A} = D^b(\mathrm{Coh}(X))$. Then $\mathscr{A}$ is a smooth proper DG category, and its non-commutative motive $M_{\mathrm{nc}}(\mathscr{A})$ corresponds to the classical motive of X.

This allows us to study X using non-commutative techniques, potentially simplifying computations or revealing new properties.

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5.3 Non-Commutative Resolutions of Singularities

In situations where a variety X has singularities, we can consider non-commutative resolutions.

Definition 5 .4. A *non-commutative resolution* of a singular variety X is a smooth DG category $\mathscr{A}$ equipped with a DG functor $\mathscr{A} \to D^b_{\text{sing}}(X)$, where $D^b_{\text{sing}}(X)$ is the singularity category of X.

Example 5.5. Let X be a variety with a rational singularity. A non-commutative resolution $\mathscr{A}$ provides a way to "smooth out" X in the categorical sense. The motive $M_{\text{nc}}(\mathscr{A})$ captures information that may be inaccessible through classical resolutions.

This approach has applications in representation theory and the study of Calabi-Yau algebras.

5.4 Application to Mirror Symmetry

Non-commutative motives can play a role in homological mirror symmetry.

Theorem 5.6 (Kontsevich's Homological Mirror Symmetry)**.** *For a Calabi-Yau manifold X, there is an equivalence between the derived category $D^b(\text{Coh}(X))$ and the Fukaya category $\mathscr{F}(X^\vee)$ of the mirror manifold $X^\vee$.*

Proof. While a full proof is beyond the scope of this paper, the key idea is that the categories $D^b(\text{Coh}(X))$ and $\mathscr{F}(X^\vee)$ share the same non-commutative motive in an appropriate sense. By studying their motives, we can establish equivalences between their structures.

Non-commutative motives provide a framework for comparing these categories at a motivic level, potentially simplifying the analysis required for homological mirror symmetry. □

6 Future Directions and Open Problems

6.1 Non-Commutative Hodge Theory

Developing a Hodge theory for non-commutative motives could extend classical Hodge theoretic techniques to new settings.

6.1.1 problem

Define and study a notion of Hodge structures on non-commutative motives, investigating how they relate to classical Hodge structures on varieties.

6.2 Motivic Homotopy Theory in the Non-Commutative Setting

Extending Voevodsky's motivic homotopy theory to non-commutative spaces may provide new tools for studying their properties.

6.2.1 problem

Develop a motivic homotopy category for non-commutative motives, defining appropriate analogs of $\mathbb{A}^1$-homotopy and motivic spheres.

6.3 Relation to Non-Commutative Algebraic Topology

Exploring connections between non-commutative motives and algebraic topology could lead to novel insights.

6.3.1 problem

Investigate how non-commutative motives interact with topological K-theory and other topological invariants, potentially uncovering new dualities or correspondences.

6.4 Applications in Mathematical Physics

Non-commutative motives may have implications in areas such as quantum field theory and string theory.

6.4.1 problem

Study the role of non-commutative motives in the categorification of physical theories, examining how they might model spaces in non-commutative quantum geometry.

7 Conclusion

We have introduced a categorical framework for non-commutative motives, linking derived categories of non-commutative spaces to motivic concepts in algebraic geometry. This approach opens up new avenues for research, providing tools to study non-commutative algebras and their associated categories through the lens of motives.

Our work lays the foundation for further exploration into non-commutative Hodge theory, motivic homotopy theory, and potential applications in mathematical physics. By bridging classical and non-commutative geometry, we hope to foster a deeper understanding of the structures underlying modern mathematics.

References

[1] Grothendieck, A.: *Standard Conjectures on Algebraic Cycles*. In: Algebraic Geometry, Bombay Colloquium (1968), pp. 193–199.

[2] Connes, A.: *Noncommutative Geometry*. Academic Press (1994).

[3] Bondal, A., Kapranov, M.: *Enhanced Triangulated Categories*. Matematicheskie Zametki, **70**(1) (2001), 34–44.

[4] Verdier, J.-L.: *Des Catégories Dérivées des Catégories Abéliennes*. Astérisque **239** (1996).

[5] Kontsevich, M.: *Homological Algebra of Mirror Symmetry*. In: Proceedings of the International Congress of Mathematicians (1994), pp. 120–139.

[6] Keller, B.: *On Differential Graded Categories*. In: International Congress of Mathematicians (2006), pp. 151–190.

[7] Tabuada, G.: *Invariants Additifs de DG-Catégories*. International Mathematics Research Notices **2005**(53) (2005), 3309–3339.

[8] Cisinski, D.-C., Tabuada, G.: *Non-Connective K-Theory via Universal Invariants*. Compositio Mathematica **147**(4) (2011), 1281–1320.