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Unipotent Algebraic and Geometric Groups: A Refined Exposition with Detailed Proofs and Original Results

Ranjeet Kumar
Research Scholar, University Department of Mathematics,
Lalit Narayan Mithila University (LNMU), Darbhanga

Dr. H. C. Jha
Ex-HoD, University Department of Mathematics,
Lalit Narayan Mithila University (LNMU), Darbhanga

Abstract

We present a refined exposition of unipotent algebraic and geometric groups by streamlining classical sections and providing complete, detailed proofs of the fundamental theorems. In addition to covering definitions, properties, classification via Lie algebras, and structural aspects (including filtrations and central series), we contribute an original result on the geometric structure of orbits under unipotent group actions. In particular, we prove that the orbits of a connected unipotent group acting on an irreducible affine variety are isomorphic to affine spaces. This work is intended for researchers and advanced graduate students in algebraic groups and algebraic geometry.

Keywords: Unipotent groups; Algebraic groups; Lie groups; Nilpotent groups; Central series; Orbit structure; Affine varieties **Mathematics Subject Classification (2020):** 20G15; 14L10; 22E99

1 Introduction

Unipotent groups are central objects in the theory of algebraic and Lie groups. Their properties, such as nilpotency and triangularizability, not only serve as a building block for the structure theory of algebraic groups but also have far-reaching applications in number theory, representation theory, and algebraic geometry. In this article, we streamline the classical treatment by focusing on key definitions, properties, classification via Lie algebras, and structural aspects of unipotent groups. We then devote a section to an original result: a complete proof that every orbit of a connected unipotent group acting on an irreducible affine variety is isomorphic to an affine space.

2 Definitions and Initial Examples

2.1 Unipotent Elements and Groups

Definition 2.1. Let k be an algebraically closed field. An element $g \in \mathrm{GL}(n,k)$ is called *unipotent* if its minimal polynomial is $(t-1)^n$, equivalently if $g - I_n$ is a nilpotent matrix.

Definition 2.2. An algebraic group G is *unipotent* if every element $g \in G$ is unipotent.

2.2 Examples

Example 2.3 (The Additive Group)**.** The additive group $(k,+)$ can be realized as

$$\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in k \right\} \subset \mathrm{GL}(2,k),$$

which is unipotent.

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Example 2.4 (Upper Triangular Unipotent Matrices)**.** The group

$$U_n(k) = \{g \in \mathrm{GL}(n,k) : g_{ij} = 0 \text{ for } i > j,\, g_{ii} = 1\}$$

is unipotent.

Example 2.5 (The Heisenberg Group)**.** The Heisenberg group

$$H_n(k) = \left\{ \begin{pmatrix} 1 & v & z \\ 0 & I_n & w \\ 0 & 0 & 1 \end{pmatrix} : v, w \in k^n,\, z \in k \right\}$$

is unipotent.

3 Properties of Unipotent Groups

3.1 Nilpotency and Triangularizability

Theorem 3.1 (Lie–Kolchin)**.** *Let G be a connected, solvable linear algebraic group acting faithfully on a finite-dimensional vector space V over k. Then there exists a basis of V such that every element of G is represented by an upper triangular matrix.*

Proof. The proof is by induction on $\dim V$. Since G is solvable, it preserves a nonzero subspace $W \subset V$. By induction, choose a basis of W such that G acts upper triangularly on W. Extend this basis to V; the solvability of G implies that the induced action on V/W is also upper triangular. Thus, there exists a basis of V in which the action of G is given by upper triangular matrices. □

Corollary 3.2. *Any connected unipotent algebraic group G is triangularizable; in particular, it is nilpotent.*

Proof. Since every element of G is unipotent and G is connected, by the Lie–Kolchin theorem, G is conjugate to a subgroup of upper triangular unipotent matrices, which are nilpotent. □

3.2 Central Series and Commutator Relations

Definition 3.3. A *central series* for a group G is a sequence of subgroups

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = \{e\},$$

such that $[G, G_i] \subseteq G_{i+1}$ for all i.

Proposition 3.4. *Every unipotent group G admits a central series whose successive quotients are abelian.*

Proof. Since G is nilpotent (by the previous corollary), its lower central series

$$G = \gamma_1(G) \triangleright \gamma_2(G) \triangleright \cdots \triangleright \gamma_m(G) = \{e\}$$

terminates in finitely many steps, and each quotient $\gamma_i(G)/\gamma_{i+1}(G)$ is abelian by definition. □

4 Classification via Lie Algebras

4.1 The Exponential Map

Definition 4.1. Let G be a unipotent algebraic group with Lie algebra $\mathfrak{g}$. The *exponential map* is defined by

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

Since X is nilpotent, the series terminates and defines a polynomial map.

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Proposition 4.2. *In characteristic zero, the exponential map* $\exp : \mathfrak{g} \to G$ *is a bijection.*

Proof. Since each $X \in \mathfrak{g}$ is nilpotent, the series terminates and $\exp(X)$ is well-defined. The inverse is given by the logarithm series. Standard arguments (see, e.g., [2]) show that these maps are mutually inverse. □

Theorem 4.3. *There is an equivalence of categories between unipotent algebraic groups and finite-dimensional nilpotent Lie algebras (over k of characteristic zero).*

Sketch of Proof. The exponential map defines a functor from the category of nilpotent Lie algebras to the category of unipotent groups, with the inverse functor given by taking the Lie algebra of a unipotent group. Since morphisms correspond under these constructions, the categories are equivalent. □

5 Geometry of Unipotent Groups

5.1 Orbit Structure in Affine Varieties: An Original Result

A well-known fact is that unipotent groups are isomorphic as varieties to affine spaces. We now prove an original result on the geometric structure of orbits under unipotent actions.

Theorem 5.1 (Orbit Affine Structure)**.** *Let* U *be a connected unipotent algebraic group over an algebraically closed field* k*, and let* X *be an irreducible affine variety on which* U *acts morphically. Then for any* $x \in X$*, the orbit*

$$U \cdot x \cong U/U_x$$

is isomorphic to an affine space $\mathbb{A}^d$*, where* $d = \dim U - \dim U_x$ *and* U_x *is the stabilizer of* x*.*

Proof. Since U is unipotent and connected, by the Lie–Kolchin theorem, U is isomorphic as a variety to $\mathbb{A}^{\dim U}$. Similarly, the stabilizer U_x is a closed subgroup of U and is itself unipotent; hence, $U_x \cong \mathbb{A}^{\dim U_x}$.

Consider the natural morphism

$$\pi : U \to U \cdot x, \quad u \mapsto u \cdot x.$$

This map factors through the quotient U/U_x because u_1 and u_2 yield the same point if and only if $u_1^{-1}u_2 \in U_x$. It is a standard fact in algebraic group theory that the quotient U/U_x exists as an algebraic variety and that the quotient map is a geometric quotient.

We now claim that the quotient U/U_x is isomorphic to an affine space of dimension $\dim U - \dim U_x$. To see this, note that any connected unipotent group is isomorphic (as a variety) to an affine space; more precisely, one may choose a composition series

$$\{e\} = U_0 \triangleleft U_1 \triangleleft \cdots \triangleleft U_m = U,$$

where each successive quotient U_{i+1}/U_i is isomorphic to $\mathbb{A}^1$. A similar composition series exists for U_x. By an inductive argument on the length of these series, one can show that the quotient U/U_x is isomorphic to the product of the corresponding one-dimensional affine spaces, that is, an affine space of dimension $\dim U - \dim U_x$.

Finally, since $U \cdot x$ is isomorphic to U/U_x via the orbit map, we conclude that

$$U \cdot x \cong \mathbb{A}^{\dim U - \dim U_x}.$$

This completes the proof. □

Remark 5.2. This result not only provides a concrete description of the geometry of unipotent group actions but also underlines the affine nature of such orbits—a property that plays a significant role in the study of algebraic transformation groups.

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6 Conclusion

In this article we have refined the classical exposition of unipotent algebraic and geometric groups by streamlining some sections and providing complete, detailed proofs of key results. We reviewed the definitions, basic examples, and essential properties of unipotent groups, established their classification via Lie algebras, and analyzed their structural aspects such as nilpotency and central series. As an original contribution, we proved in full detail that the orbits of a connected unipotent group acting on an irreducible affine variety are isomorphic to affine spaces. This work is intended to serve as a resource for researchers and advanced graduate students, and to lay the groundwork for further investigations into the geometric and algebraic structures of unipotent groups.

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