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Analysing Bicomplex Matrices in Mathematica

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Abstract— A bicomplex number is a composition of two complex numbers or four real numbers. However, a Bicomplex Matrix or a BC-Matrix is a $\boldsymbol{m} \times \boldsymbol{n}$ array of bicomplex numbers. Recently, studies on BC-matrices have caught interest of mathematicians where determinant and other properties have been explored. In present study on BC-matrices, mathematical computation software Mathematica has been applied and various examples and counterexamples are provided to substantiate the results. Step by step commands were given to find the determinant and Eigen values of a $\boldsymbol{n} \times \boldsymbol{n}$ BC-matrix for which rigorous calculations are needed, if the matrix is of higher order. In a particular case, Eigen values of a BC-matrix are obtained using single command for which we can verify the relationships between determinant, trace and the product of eigen values of the matrix. Eigen values by usual method (i.e., by solving the characteristic equation) have also been obtained and differences have been shown in both the outcomes. Unimodular Bicomplex matrices have been defined and an example for the same has been given and analyzed. Bicomplex numbers have found deep rooted applications in many areas of sciences, so this development may help in their applications with accuracy and much less time.

Keywords- Bicomplex numbers, Bicomplex Matrix, BC-Matrix, Mathematica
Mathematics Subject Classification: 68U99, 1533, 65F15, 97N80

I. Introduction

Highlights:

- Analysis of BC-matrices in Mathematica with much lesser computation time.
- Finding determinant, Eigen Values of these matrices with the help of Mathematica commands.
- Unimodular BC-matrices.

Notations:

The symbol $\mathbb{C}_0$ is for real numbers, $\mathbb{C}_1$ for complex numbers and $\mathbb{C}_2$ for bicomplex numbers. First idempotent part of a bicomplex set X, when the complex space is $\mathbb{C}_1(i_1)$, is denoted by 1_A. Second idempotent part of a bicomplex set X, when the complex space is $\mathbb{C}_1(i_1)$, is denoted by 2_A. First idempotent part of a bicomplex set X, when the complex space is $\mathbb{C}_1(i_2)$, is denoted by A_1. Second idempotent part of a bicomplex set X, when the complex space is $\mathbb{C}_1(i_2)$, is denoted by A_2.

Before defining bicomplex numbers, let us see what quaternions are.

1.1 Quaternions

Quaternions were defined by Irish mathematician, Sir William Rowan Hamilton in 1843. A quaternion is defined as a four-dimensional number given by

$$x_1 + ix_2 + jx_3 + kx_4,$$

where $x_1, x_2, x_3, x_4 \in C_0$ and $i^2, j^2, k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$. He defined quaternion as the quotient of two vectors.

The quaternions were the first non-commutative division algebra to be discovered.

1.2 Bicomplex Space

$$\mathbb{C}_2 = \{x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 : x_1, x_2, x_3, x_4 \in \mathbb{C}_0\} \text{ where } i_1^2 = i_2^2 = -1, i_1i_2 = i_2i_1$$

or

$$\mathbb{C}_2 = \{z_1 + i_2z_2 : z_1, z_2 \in \mathbb{C}_1\}$$

represents the set of bicomplex numbers.

Bicomplex numbers (defined by Corrado Segre [7] in 1892) and quaternions both are real four-dimensional spaces but $\mathbb{C}_2$ is a commutative space. In many ways bicomplex numbers generalize complex numbers more closely and more accurately than quaternions do. But commutativity is gained at a price. It is a well-known fact that the ring of bicomplex numbers is not a field, since zero divisors arise to prevent such a possibility. Some important applications of bicomplex algebra may be seen in [2]. Recently, a lot of work has been done in bicomplex sequence spaces [10, 11], bicomplex functional analysis [1], bicomplex matrices [14], bicomplex duals [13], bicomplex modules [12].

1.3 Differences Between the Algebraic Structure of $\mathbb{C}_2$ and $\mathbb{C}_1$
Singular elements other than zero, exist in $\mathbb{C}_2$. In complex space we have only two idempotent elements, but in $\mathbb{C}_2$, there are nontrivial idempotent elements. Non-trivial zero divisors also exist in bicomplex space.

1.4 Idempotent Elements in $\mathbb{C}_2$
Besides 0 and 1 there are two other idempotent elements in $\mathbb{C}_2$, given by

$$e_1 = \frac{1 + i_1 i_2}{2}, e_2 = \frac{1 - i_1 i_2}{2}.$$

1.5 Idempotent Representation [9]
There are two types of idempotent representation of a bicomplex number.

First idempotent representation

$$\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$$
$$= (x_1 + i_1 x_2) + i_2(x_3 + i_1 x_4) = z_1 + i_2 z_2$$
$$= (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2 t)e_2 = 1_\xi e_1 + 2_\xi e_2$$

here, $1_\xi, 2_\xi \in C_1(i_1)$ are the idempotent parts of the bicomplex number ξ, when we consider the complex space with imaginary unit i_1.

Second idempotent representation:

$$\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$$
$$= (x_1 + i_2 x_3) + i_1(x_2 + i_2 x_4) = w_1 + i_1 w_2$$
$$= (w_1 - i_2 w_2)e_1 + (w_1 + i_2 w_2)e_2 = \xi_1 e_1 + \xi_2 e_2$$

here, $\xi_1, \xi_2 \in C_1(i_2)$ are the idempotent parts of the bicomplex number ξ, when we consider the complex space with imaginary unit i_2. Idempotent representations play an important role in the analysis of bicomplex numbers as all the mathematical operations can be performed component-wise with this representation. For detailed study of bicomplex numbers one can refer to the book by Price [6].

II. Bicomplex Matrices and their analysis in Mathematica

2.1 Bicomplex Matrix - Definition
A matrix $A = [\xi_{mn}]_{m\times n}$ whose entries are from the set of bicomplex numbers $\mathbb{C}_2$ is called a bicomplex matrix.
I.e.,

$$A = \begin{pmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{m1} & \xi_{m2} & \cdots & \xi_{mn} \end{pmatrix}_{m\times n}, \xi_{pq} \in \mathbb{C}_2, 1 \le p \le m, 1 \le q \le n$$

Bicomplex matrices have been studied by many authors. A few authors have studied the structure of bicomplex matrices. Alpay et al. has given a chapter on Bicomplex functions and Matrices in [1]. Jogendra in [3] and 4] have studied about the determinant and eigen values of bicomplex matrices. William and Rebecca in [13] have studied Jordan Forms, Invariant Subspace Lattice Diagrams, and Compact Operators.

2.2 Methodology
For the analysis of bicomplex matrices Mathematica version 11.3 has been used in this study. It is a software system with inbuilt libraries to be applied in different domains of technical computing. The built-in functions and commands in Mathematica permit analysis of various types of data from fields such as "machine learning, symbolic computation, statistics, data manipulation, network analysis, time series analysis, optimization, plotting functions and various types of data, implementation of algorithms etc".
Note 2.2.1. (1) Here we are replacing i_1 by I and i_2 by J, as per the symbols defined in Mathematica. (2) In Mathematica, to execute any command we press "shift + enter" and get the output.

2.3 Determinant of a BC-matrix
We know that only square matrices can have determinant, let's consider a square BC-matrix $A = [\xi_{ij}]_{n\times n} \in C_2^{n\times n}$. Then

$$\det(A) = \det(1_A)e_1 + \det(2_A)e_2 \text{ or}$$
$$\det(A) = \det(A_1)e_1 + \det(A_2)e_2$$

Computing determinat of a BC-matrix
Step 1: $\det\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right]$ (press shift + enter)
Step 2: Simplify the expression obtained in Step 1 for second imaginary unit J by
Simplify [expression, $j^2 == -1$] (press shift + enter)

2.4 Unimodular Bicomplex Matrix

Definition 2.4.1. A square bicomplex matrix having determinant +1 or -1 . The inverse of a unimodular matrix is also unimodular.

Example of a unimodular BC-matrix:

$$\begin{pmatrix} (1 + I * J)/2 & (1 - I * J)/2 \\ (2 + 4I * J)/2 & (1 + I * J)/2 \end{pmatrix}$$

let's verify it by finding its determinant.

$$\det\left[\begin{pmatrix} (1 + I * J)/2 & (1 - I * J)/2 \\ (2 + 4I * J)/2 & (1 + I * J)/2 \end{pmatrix}\right]$$

$$\text{Out}[\cdot] = -\frac{1}{4} - \frac{5J^2}{4}$$

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Since Mathematica does not take the symbol J as the second imaginary unit i_2, we must simplify the above expression by replacing J^2 by -1.

$$\text{In}[\cdot]:=\ \text{Simplify}\left[-\frac{1}{4}-\frac{5J^2}{4}, J^2=-1\right]$$

$$\text{Out}\ [\cdot]=1$$

2.5 Computing Inverse of a BC-matrix

Step 1: Inverse $\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\right]$ (press shift + enter)

Step 2: Write the expression obtained in step 1 in matrix form by

MatrixForm[{expression }] (press shift + enter)

Step 3: Simplify the expression obtained in Step 2 for second imaginary unit J by

Simplify[expression, $j^2==-1$] (press shift + enter)

Example 2.5.1. Now, lets find the inverse of a BC-matrix with an example.

Consider the matrix $B=\begin{pmatrix} (1+I*J)/2 & (1-I*J)/2 \\ (2+4I*J)/2 & (1+I*J)/2\end{pmatrix}$.

$$\text{In}[\cdot]:=\ \text{Inverse}\left[\begin{pmatrix} (1+i*J)/2 & (1-i*J)/2 \\ (2+4I*J)/2 & (1+i*J)/2\end{pmatrix}\right]$$

$$\text{Out}\ [\cdot]=\left\{\left\{\frac{1+\mathrm{i}j}{2\left(-\frac{1}{4}-\frac{5J^2}{4}\right)},\frac{-1+\mathrm{i}j}{2\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}\right\},\left\{\frac{-2-4\mathrm{i}J}{2\left(-\frac{1}{4}-\frac{5J^2}{4}\right)},\frac{1+\mathrm{i}j}{2\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}\right\}\right\}$$

$$\text{In}[\cdot]:=\ \text{MatrixForm}\left[\left[\left\{\left\{\frac{1+\mathrm{i}J}{2\left(-\frac{1}{4}-\frac{5J^2}{4}\right)},\frac{-1+\mathrm{i}J}{2\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}\right\},\left\{\frac{-2-4\mathrm{i}J}{2\left(-\frac{1}{4}-\frac{5J^2}{4}\right)},\frac{1+\mathrm{i}J}{2\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}\right\}\right\}\right]\right]$$

$$\left(\frac{1+\mathrm{i}}{2\left(-\frac{1}{4}-\frac{5}{4}\right.}\left(\frac{-1+\mathrm{i}J}{4}\right) \quad \frac{1+\frac{5}{4}J^2}{2\left(-\frac{1}{4}-\frac{5}{4}J^2\right)}\right)\}$$

(For verification, lets find the product of the matrix B and its inverse)

$$\text{In}[\cdot]:=\begin{pmatrix} (1+I*J)/2 & (1-I*J)/2 \\ (2+4I*J)/2 & (1+I*J)/2\end{pmatrix}$$

$$.\begin{pmatrix} \frac{1+\mathrm{i}J}{2\left(-\frac{1}{4}-\frac{5.J^2}{4}\right)} & \frac{-1+\mathrm{i}J}{2\left(-\frac{1}{4}-\frac{5.J^2}{4}\right)} \\ \frac{-2-4\mathrm{i}J}{2\left(-\frac{1}{4}-\frac{5.J^2}{4}\right)} & \frac{1+\mathrm{i}J}{2\left(-\frac{1}{4}-\frac{5.J^2}{4}\right)}\end{pmatrix}$$

$$\text{Out}\ [\cdot]=\left\{\left\{\frac{(1+\mathrm{i}J)^2}{4\left(-\frac{1}{4}-\frac{5.J^2}{4}\right)}+\frac{(1-\mathrm{i}J)(-2-4\mathrm{i}J)}{4\left(-\frac{1}{4}-\frac{5.J^2}{4}\right)},\frac{(1-\mathrm{i}J)(1+\mathrm{i}J)}{4\left(-\frac{1}{4}-\frac{5.J^2}{4}\right)}+\frac{(-1+\mathrm{i}J)(1+\mathrm{i}J)}{4\left(-\frac{1}{4}-\frac{5.J^2}{4}\right)}\right\},\right.$$

$$\left.\left\{\frac{(1+\mathrm{i}j)(-2-4\mathrm{i}J)}{4\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}+\frac{(1+\mathrm{i}j)(2+4\mathrm{i}J)}{4\left(-\frac{1}{4}-\frac{5J^2}{4}\right)},\frac{(1+\mathrm{i}j)^2}{4\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}+\frac{(-1+\mathrm{i}j)(2+4\mathrm{i}J)}{4\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}\right\}\right\},$$

$$\text{In}\ [\cdot]:=\ \text{Simplify}\left[\left\{\left\{\frac{(1+\mathrm{i}J)^2}{4\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}+\frac{(1-\mathrm{i}J)(-2-4\mathrm{i}J)}{4\left(-\frac{1}{4}-\frac{5J^2}{4}\right)},\frac{(1-\mathrm{i}J)(1+\mathrm{i}J)}{4\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}+\frac{(-1+\mathrm{i}J)(1+\mathrm{i}J)}{4\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}\right\},\right.\right.$$

$$\left.\left.\left\{\frac{(1+\mathrm{i}J)(-2-4\mathrm{i}J)}{4\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}+\frac{(1+\mathrm{i}J)(2+4\mathrm{i}J)}{4\left(-\frac{1}{4}-\frac{5J^2}{4}\right)},\frac{(1+\mathrm{i}J)^2}{4\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}+\frac{(-1+\mathrm{i}J)(2+4\mathrm{i}J)}{4\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}\right\}\right\}, J^2=-1\right]$$

$$\text{Out}\ [\cdot]=\{\{1,0\},\{0,1\}\}$$

which is an identity matrix.

Let the above inverse is denoted by matrix W, i.e., $W=\begin{pmatrix} \frac{1+\mathrm{i},J}{2\left(-\frac{1}{4}-\frac{5J^2}{4}\right)} & \frac{-1+\mathrm{i}J}{2\left(-\frac{1}{4}-\frac{5J^2}{4}\right)} \\ \frac{-2-4\mathrm{i}J}{2\left(-\frac{1}{4}-\frac{5J^2}{4}\right)} & \frac{1+\mathrm{i},J}{2\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}\end{pmatrix}$.

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Note 2.5.1. The above matrix W also serves as another example of unimodular BC-matrix. Basically, the inverse of a unimodular BC-matrix is another unimodular matrix. To verify, lets find the determinant of matrix W.

$$\text{In}[\cdot] := \det\left[\left\{\left\{\frac{1+\mathrm{i}J}{2\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}, \frac{-1+\mathrm{i}J}{2\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}\right\}, \left\{\frac{-2-4\mathrm{i}J}{2\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}, \frac{1+\mathrm{i}J}{2\left(-\frac{1}{4}-\frac{5J^2}{4}\right)}\right\}\right\}\right]$$

$$\text{Out}[\cdot] = -\frac{4}{(1+5J^2)^2} - \frac{20J^2}{(1+5J^2)^2}$$

$$\text{In}[\cdot] := \text{ Simplify }\left[-\frac{4}{(1+5J^2)^2} - \frac{20J^2}{(1+5J^2)^2}, J^2 = -1\right]$$

$$\text{Out}[\cdot] = 1$$

(Hence, we conclude that $\begin{pmatrix}(1+i*J)/2 & (1-i*J)/2\\ (2+4i*J)/2 & (1+i*J)/2\end{pmatrix}$ is a UNIMODULAR BC-matrix.)

2.6 Computing Transpose of a BC-Matrix ' Y '

$$\text{In}[\cdot] := \text{ Transpose }\left[\begin{pmatrix}(1+I*J)/2 & (2-4I*J)/2\\ (1+I*J)/2 & (1+I*J)/2\end{pmatrix}\right]$$

$$\text{Out}[\cdot] = \left\{\left\{\frac{1}{2}(1+\mathrm{i}J), \frac{1}{2}(1+\mathrm{i}J)\right\}, \left\{\frac{1}{2}(2-4\mathrm{i}J), \frac{1}{2}(1+\mathrm{i}J)\right\}\right\}$$

$$\text{In}[\cdot] := \text{ MatrixForm }\left[\left\{\left\{\frac{1}{2}(1+\mathrm{i}J), \frac{1}{2}(1+\mathrm{i}J)\right\}, \left\{\frac{1}{2}(2-4\mathrm{i}J), \frac{1}{2}(1+\mathrm{i}J)\right\}\right\}\right]$$

$$\text{Out}[\cdot]//\text{ MatrixForm } = \begin{pmatrix}\frac{1}{2}(1+\mathrm{i}J) & \frac{1}{2}(1+\mathrm{i}J)\\ \frac{1}{2}(2-4\mathrm{i}J) & \frac{1}{2}(1+\mathrm{i}J)\end{pmatrix}$$

Computing Determinant of the Transpose of BC-Matrix 'Y'

Example 2.6.1.

$$\text{In}[\cdot] := \det\left[\left\{\left\{\frac{1}{2}(1+\mathrm{i}J), \frac{1}{2}(1+\mathrm{i}J)\right\}, \left\{\frac{1}{2}(2-4\mathrm{i}J), \frac{1}{2}(1+\mathrm{i}J)\right\}\right\}\right]$$

$$\text{Out}[\cdot] = -\frac{1}{4} + \mathrm{i}J - \frac{5J^2}{4}$$

$$\text{In}[\cdot] := \text{ Simplify }\left[-\frac{1}{4} + \mathrm{i}J - \frac{5J^2}{4}, J^2 = -1\right]$$

$$\text{Out}[\cdot] = 1 + \mathrm{i}J$$

(with above example, we see that determinant of a BC-matrix is same as determinant of its transpose.)

Example 2.6.2.

$$\text{In}[\cdot] := \det\left[\begin{pmatrix}(1+I*J)/2 & 1/2\\ (1+I*J)/2 & (1+I*J)/2\end{pmatrix}\right]$$

$$\text{Out}[\cdot] = \frac{\mathrm{i}J}{4} - \frac{J^2}{4}$$

$$\text{In}[\cdot] := \partial_J\left(\frac{\mathrm{i}J}{4} - \frac{J^2}{4}\right)$$

$$\text{Out}[\cdot] = \frac{\mathrm{i}}{4} - \frac{J}{2}$$

$$\text{In}[\cdot] := \partial_J\left(\frac{\mathrm{i}}{4} - \frac{J}{2}\right)$$

$$\text{Out}[\cdot] = -\frac{1}{2}$$

(Note: We see that computing the double J-derivate (or computing the J-derivative twice) of the determinant of a BC-matrix $\begin{pmatrix}(1+I*J)/2 & 1/2\\ (1+I*J)/2 & (1+I*J)/2\end{pmatrix}$ gives $-1/2$)

$$\text{In}[\cdot] := \text{ Simplify }\left[\frac{\mathrm{i}J}{4} - \frac{1}{4}\right.$$

$$\text{Out}[\cdot] = \frac{1}{4}(1+\mathrm{i}J)$$

$$\text{In}[\cdot] := \partial_J\left(\frac{1}{4}(1+\mathrm{i}J)\right)$$

$$\text{Out}[\cdot] = \frac{\mathrm{i}}{4}$$

Note 2.6.1. If we first simplify the determinant for J and then find the J-derivative gives a complex number.

Example 2.6.3. Consider another BC-matrix $\begin{pmatrix}(3+I*J)/2 & 1/5\\ (1+I*J)/2 & (1+I*J)/2\end{pmatrix}$,

$$\text{In}[\cdot] := \det\left[\begin{pmatrix} (3+I*J)/2 & 1/5 \\ (1+I*J)/2 & (1+I*J)/2 \end{pmatrix}\right]$$

$$\text{Out}[\cdot] = \frac{13}{20} + \frac{9iJ}{10} - \frac{J^2}{4}$$

$$\text{In}[\cdot] := \text{Simplify}\left[\frac{13}{20} + \frac{9iJ}{10} - \frac{J^2}{4}, J^2 == 1\right]$$

$$\text{Out}[\cdot] = \frac{9}{10} i(-i + J)$$

$$\text{In}[\cdot] := \partial_J\left(\frac{9}{10} i(-i + J)\right)$$

$$\text{Out}[\cdot] = \frac{9i}{10}$$

(Now lets find J-derivative of the determinant)

$$\text{In}[\cdot] := \partial_J\left(\frac{13}{20} + \frac{9iJ}{10} - \frac{J^2}{4}\right)$$

$$\text{In}[\cdot] = \frac{9i}{10} - \frac{J}{2}$$

$$\text{In}[\cdot] := \partial_J\left(\frac{9i}{10} - \frac{J}{2}\right)$$

$$\text{Out}[\cdot] = -\frac{1}{2}$$

Observation 2.6.1. Simplifying the determinant of a BC-matrix for J and then finding the J-derivative is not same as finding the J derivatives of the determinant.

2.7 Examples of Singular BC-Matrix

Example 2.7.1.

$$\text{In}[\cdot] := \det\left[\left(\begin{pmatrix} 3*\left(\frac{1+I*J}{2}\right) & 5*\left(\frac{1-I*J}{2}\right) \\ 1*\left(\frac{1+I*J}{2}\right) & 7*\left(\frac{1-I*J}{2}\right) \end{pmatrix}\right)\right]$$

$$\text{Out}[\cdot] = 4 + 4J^2$$

$$\text{In}[\cdot] := \text{Simplify}\,[4 + 4J', J' == -1]$$

$$\text{Out}[\cdot] = 0$$

Since, $\det(A) = \det\begin{pmatrix} 3*\left(\frac{1+I*J}{2}\right) & 5*\left(\frac{1-I*J}{2}\right) \\ 1*\left(\frac{1+I*J}{2}\right) & 7*\left(\frac{1-I*J}{2}\right) \end{pmatrix} = 0$,

therefore A is a singular bicomplex matrix.

Example 2.7.2.

$$\text{In}[\cdot] := \det\left[\left(\begin{pmatrix} \left(\frac{1+I*J}{2}\right) & \left(\frac{1-I*J}{2}\right) \\ \left(\frac{1+I*J}{2}\right) & \left(\frac{1-I*J}{2}\right) \end{pmatrix}\right)\right]$$

$$\text{Out}[\cdot] = 0$$

Example 2.7.3 (Generalized example of singular bicomplex matrix).

$$\text{In}[\cdot] := \det\left[\left(\begin{pmatrix} a*\left(\frac{1+I*J}{2}\right) & b*\left(\frac{1-I*J}{2}\right) \\ c*\left(\frac{1+I*J}{2}\right) & d*\left(\frac{1-I*J}{2}\right) \end{pmatrix}\right)\right]$$

$$\text{Out}[\cdot] = -\frac{bc}{4} + \frac{ad}{4} - \frac{1}{4}bcJ^2 + \frac{1}{4}adJ^2$$

$$\text{In}[\cdot] := \text{Simplify}\left[-\frac{bc}{4} + \frac{ad}{4} - \frac{1}{4}bcJ^2 + \frac{1}{4}adJ^2, J^2 = -1\right]$$

$$\text{Out}[\cdot] = 0$$

Hence $\begin{pmatrix} a*\left(\frac{1+I*J}{2}\right) & b*\left(\frac{1-I*J}{2}\right) \\ c*\left(\frac{1+I*J}{2}\right) & d*\left(\frac{1-I*J}{2}\right) \end{pmatrix}$ is a singular matrix for any a, b, c, d real numbers.

Notice that a, b, c, d may be from complex space or from bicomplex space, this matrix will always be a singular matrix.

For example, $\begin{bmatrix} J*\left(\frac{1-I*J}{2}\right) & I*\left(\frac{1-I*J}{2}\right) \\ I*\left(\frac{1+I*J}{2}\right) & J*\left(\frac{1+I*J}{2}\right) \end{bmatrix}$ and $\begin{bmatrix} J*\left(\frac{1-I*J}{2}\right) & I*J*\left(\frac{1-I*J}{2}\right) \\ I*\left(\frac{1+I*J}{2}\right) & J*\left(\frac{1+I*J}{2}\right) \end{bmatrix}$ are singular matrices.

$$\text{In}[\cdot] := \det\begin{bmatrix} J*\left(\frac{1-I*J}{2}\right) & I*\left(\frac{1-I*J}{2}\right) \\ I*\left(\frac{1+I*J}{2}\right) & J*\left(\frac{1+I*J}{2}\right) \end{bmatrix}$$

$$\text{In}[\cdot] = \frac{1}{4} + \frac{j^2}{2} + \frac{j^4}{4}$$

$$\text{In}[\cdot] := \text{Simplify}\left[\frac{1}{4} + \frac{J^2}{2} + \frac{J^4}{4}, J^2 = -1\right]$$

$$\text{Out}[\cdot] = 0$$

2.8 Examples of Nonsingular BC-Matrices

Example 2.8.1. Consider matrix $\begin{pmatrix} a*\left(\frac{1-I*J}{2}\right) & b*\left(\frac{1+I*J}{2}\right) \\ c*\left(\frac{1+I*J}{2}\right) & d*\left(\frac{1-I*J}{2}\right) \end{pmatrix}$

$$\text{In}[\cdot] := \det\begin{bmatrix} a*\left(\frac{1-I*J}{2}\right) b*\left(\frac{1+I*J}{2}\right) \\ c*\left(\frac{1+I*J}{2}\right) d*\left(\frac{1-I*J}{2}\right) \end{bmatrix}$$

$$\text{Out}[\cdot] = -\frac{bc}{4} + \frac{ad}{4} - \frac{1}{2}ibcJ - \frac{1}{2}iadJ + \frac{1}{4}bcJ^2 - \frac{1}{4}adJ^2$$

$$\text{In}[\cdot] := \text{Simplify}\begin{bmatrix} -\frac{bc}{4} + \frac{ad}{4} - \frac{1}{2}ibcJ - \frac{1}{2}iadJ \\ +\frac{1}{4}bcJ^2 - \frac{1}{4}adJ^2, J^2 == -1 \end{bmatrix}$$

$$\text{Out}[\cdot] = -\frac{1}{2}i(bc(-i + J) + ad(i + J))$$

Example 2.8.2. $\begin{bmatrix} a*\left(\frac{1-I*J}{2}\right) & c*\left(\frac{1+I*J}{2}\right) \\ c*\left(\frac{1+I*J}{2}\right) & a*\left(\frac{1-I*J}{2}\right) \end{bmatrix}$

$$\text{In}[\cdot] := \det\left[\begin{matrix} a*\left(\frac{1-I*J}{2}\right) & c*\left(\frac{1+I*J}{2}\right) \\ c*\left(\frac{1+I*J}{2}\right) & a*\left(\frac{1-I*J}{2}\right)\end{matrix}\right]$$

$$\text{Out}\,[\cdot] = \frac{a^2}{4} - \frac{c^2}{4} - \frac{1}{2}\text{i}a^2J - \frac{1}{2}\text{i}c^2J - \frac{a^2J^2}{4} + \frac{c^2J^2}{4}$$

$$\text{Simplify}\left[\frac{a^2}{4} - \frac{c^2}{4} - \frac{1}{2}\text{i}a^2J - \frac{1}{2}\text{i}c^2J - \frac{a^2J^2}{4} + \frac{c^2J^2}{4}\right]$$

$$\text{Out}\,[\cdot] = \frac{1}{4}\left(c^2(-\text{i}+J)^2 - a^2(\text{i}+J)^2\right)$$

$$\text{In}\,[\cdot] := \text{ Simplify}\left[\begin{matrix}\frac{1}{4}\left(c^2(-\text{i}+J)^2 - a^2(\text{i}+J)^2\right), \\ J^2 = -1\end{matrix}\right]$$

$$\text{Out}\,[\cdot] = \frac{1}{2}(a^2 - c^2 - \text{i}(a^2+c^2)J)\left(*\frac{1}{2}(a^2 - c^2 - \text{i}(a^2+c^2)J) = \frac{a^2-c^2}{2} - IJ*\frac{a^2+c^2}{2}\right)$$

2.9 Defining Idempotent Elements E1 and E2 (for Computations in Mathematica)

Lets define idempotent elements e_1 and e_2 in Mathematica and then proceed further with some more examples.

Example 2.9.1.

$$E1 = (1 + I*J)/2;$$
$$E2 = (1 - I*J)/2;$$
$$\det\left[\begin{pmatrix} E1 & E2 \\ E2 & E1\end{pmatrix}\right]$$

$$\text{Out}[\cdot] = \text{i}J$$

$$\text{In}[\cdot] := \det\left[\begin{pmatrix} -3*E1 & -5*J*E2 \\ 5*J*E1 & -i*E2\end{pmatrix}\right]$$

$$\text{Out}\,[\cdot] = \frac{3\text{i}}{4} + \left(\frac{25}{4} + \frac{3\text{i}}{4}\right)J^2 + \frac{25J^4}{4}$$

$$\text{In}[\cdot] := \text{ Simplify}\left[\frac{3\text{i}}{4} + \left(\frac{25}{4} + \frac{3\text{i}}{4}\right)J^2 + \frac{25J^4}{4}, J^2 = -1\right]$$

$$\text{Out}\,[\cdot] = 0$$

hence, $\begin{pmatrix} -3*E1 & -5*J*E2 \\ 5*J*E1 & -I*E2\end{pmatrix}$ is a singular matrix.

Example 2.9.2. $\begin{pmatrix} a*E1 & b*E2 \\ c*E1 & d*E2\end{pmatrix} \cdot \begin{pmatrix} a*E1 & b*E2 \\ c*E1 & d*E2\end{pmatrix}$
(*product of two BC-matrices*)

$$\text{Out}[\cdot] = \left\{\left\{\frac{1}{4}bc(1-\text{i}J)(1+\text{i}J) + \frac{1}{4}A^2(1+\text{i}J)^2, \frac{1}{4}bd(1-\text{i}J)^2 + \frac{1}{4}ab(1-\text{i}J)(1+\text{i}J)\right\}, \left\{\frac{1}{4}cd(1-\text{i}J)(1+\text{i}J) + \frac{1}{4}ac(1+\text{i}J)^2, \frac{1}{4}d^2(1-\text{i}J)^2 + \frac{1}{4}bc(1-\text{i}J)(1+\text{i}J)\right\}\right\}$$

$$\text{In}[\cdot] := \text{MatrixForm}\left[\left\{\left\{\frac{1}{4}bc(1-\text{i})(1+\text{i}J) + \frac{1}{4}A^2(1+\text{i}J)^2, \frac{1}{4}bd(1-\text{i}J)^2 + \frac{1}{4}ab(1-\text{i}J)(1+\text{i}J)\right\}, \left\{\frac{1}{4}cd(1-\text{i}J)(1+\text{i}J) + \frac{1}{4}ac(1+\text{i}J)^2, \frac{1}{4}d^2(1-\text{i}J)^2 + \frac{1}{4}bc(1-\text{i}J)(1+\text{i}J)\right\}\right\}\right]$$

Out[•]//MatrixForm

$$= \begin{pmatrix} \frac{1}{4}bc(1-\text{i}J)(1+\text{i}J) + \frac{1}{4}A^2(1+\text{i}J)^2 & \frac{1}{4}bd(1-\text{i}J)^2 + \frac{1}{4}ab(1-\text{i}J) & (1+\text{i}J) \\ \frac{1}{4}cd(1-\text{i}J) & (1+\text{i}J) + \frac{1}{4}ac(1+\text{i}J)^2 & \frac{1}{4}d^2(1-\text{i}J)^2 + \frac{1}{4}bc(1-\text{i}J)\end{pmatrix}(1+\text{i}J)$$

$$\text{In}[\cdot] := \det\left[\left\{\left\{\frac{1}{4}bc(1-\text{i}J)(1+\text{i}J) + \frac{1}{4}a^2(1+\text{i}J)^2, \frac{1}{4}bd(1-\text{i}J)^2 + \frac{1}{4}ab(1-\text{i}J)(1+\text{i}J)\right\}, \left\{\frac{1}{4}cd(1-\text{i}J)(1+\text{i}J) + \frac{1}{4}ac(1+\text{i}J)^2, \frac{1}{4}d^2(1-\text{i}J)^2 + \frac{1}{4}bc(1-\text{i}J)(1+\text{i}J)\right\}\right\}\right]$$

$$\text{Out}\,[\cdot] = \frac{b^2c^2}{16} - \frac{1}{8}abcd + \frac{a^2d^2}{16} + \frac{1}{8}b^2c^2J^2 - \frac{1}{4}abcdJ^2 + \frac{1}{8}a^2d^2J^2 + \frac{1}{16}b^2c^2J^4 - \frac{1}{8}abcdJ^4 + \frac{1}{16}a^2d^2J^4$$

$$\text{In}[\cdot] := \text{Simplify}\left[\begin{matrix} \frac{b^2c^2}{16} - \frac{1}{8}abcd + \frac{a^2d^2}{16} \\ +\frac{1}{8}b^2c^2J^2 - \frac{1}{4}abcdJ^2 \\ +\frac{1}{8}a^2d^2J^2 + \frac{1}{16}b^2c^2J^4 \end{matrix}\right.$$
$$\left. dJ^4 + \frac{1}{16}a^2d^2J^4, J^2 = -1\right]$$

$\text{Out}[\cdot] = 0$

hence, $\begin{pmatrix} a*E1 & b*E2 \\ c*E1 & d*E2 \end{pmatrix} \cdot \begin{pmatrix} a*E1 & b*E2 \\ c*E1 & d*E2 \end{pmatrix}$ will give a singular matrix.

2.10

Consider four bicomplex number A, B, H, P defined as follows. Show that $\det\left[\begin{pmatrix} A & H \\ B & P \end{pmatrix}\right] = \det\left[\begin{pmatrix} A & B \\ H & P \end{pmatrix}\right]$ and $\det\begin{pmatrix} A & H \\ B & P \end{pmatrix}^{-1} = 1/\det\begin{pmatrix} A & H \\ B & P \end{pmatrix}$,

$$\text{In}[\cdot] := A = 2 + 3I + 59J + I*J$$
$$B = 3 + 5I + 23J + 2I*J$$
$$H = 2 + 9I + 25J - I*J$$
$$P = 23 + 8I - 23J + 3I*J$$
$$\text{detdet}\left[\begin{pmatrix} A & H \\ H & P \end{pmatrix}\right]$$
$$\text{Out}[\cdot] = (61 + 48\text{i}) + (1186 + 99\text{i})J - (1937 - 127\text{i})J2$$
$$\text{In}[\cdot] := \text{Simplify}\left[\begin{matrix} (61 + 48\text{i}) \\ +(1186 + 99\text{i})J \\ -(1937 - 127\text{i})J^2, \\ J^2 == -1 \end{matrix}\right.$$
$$\text{Out}[\cdot] = (1998 - 79\text{i}) + (1186 + 99\text{i})J$$
$$\text{In}[\cdot] := \det\left[\begin{pmatrix} A & H \\ B & P \end{pmatrix}\right]$$
$$\text{Out}[\cdot] = (61 + 48\text{i}) + (1186 + 99\text{i})J - (1937 - 127\text{i})J^2$$
$$\text{In}[\cdot] := \text{Simplify}\left[\begin{matrix} (61 + 48\text{i}) \\ +(1186 + 99\text{i})J \\ -(1937 - 127\text{i})J^2, \\ J^2 = -1 \end{matrix}\right]$$
$$\text{Out}[\cdot] = (1998 - 79\text{i}) + (1186 + 99\text{i})J$$

therefore, $\det\left[\begin{pmatrix} A & H \\ B & P \end{pmatrix}\right] = \det\left[\begin{pmatrix} A & B \\ H & P \end{pmatrix}\right]$.

Now,

$$\text{In}[\cdot] := \text{Inverse}\left[\begin{pmatrix} A & B \\ H & P \end{pmatrix}\right]$$

$$\text{Out}[\cdot] = \left\{\left\{\frac{(23 + 8\text{i}) - (23 - 3\text{i})J}{(61 + 48\text{i}) + (1186 + 99\text{i})J - (1937 - 127\text{i})J^2}, \frac{(-3 - 5\text{i}) - (23 + 2\text{i})J}{(61 + 48\text{i}) + (1186 + 99\text{i})J - (1937 - 127\text{i})J^2}\right\}, \left\{\frac{(-2 - 9\text{i}) - (25 - \text{i})J}{(61 + 48\text{i}) + (1186 + 99\text{i})J - (1937 - 127\text{i})J^2}, \frac{(2 + 3\text{i}) + (59 + \text{i})J}{(61 + 48\text{i}) + (1186 + 99\text{i})J - (1937 - 127\text{i})J^2}\right\}\right\}$$

$$\text{In}[\cdot] := \det[\%104]$$

$$\text{Out}[\cdot] = -\frac{61 + 48\text{i}}{\left((48 - 61\text{i}) + (99 - 1186\text{i})J + (127 + 1937\text{i})J^2\right)^2} - \frac{(1186 + 99\text{i})J}{\left((48 - 61\text{i}) + (99 - 1186\text{i})J + (127 + 1937\text{i})J^2\right)^2} + \frac{(1937 - 127\text{i})J^2}{\left((48 - 61\text{i}) + (99 - 1186\text{i})J + (127 + 1937\text{i})J^2\right)^2}$$

$$\text{In}[\cdot] := \text{Simplify}\left[-\frac{61 + 48\text{i})}{\left(((48 - 61\text{i})) + (99 - 1186\text{i}))j + (127 + 1937\text{i}))J^2\right)^2} - \frac{(1186 + 99\text{i})J}{\left((48 - 61\text{i}) + (99 - 1186\text{i})J + (127 + 1937\text{i})J^2\right)^2} + \frac{(1937 - 127\text{i})J^2}{\left((48 - 61\text{i}) + (99 - 1186\text{i})J + (127 + 1937\text{i})J^2\right)^2}, J^2 = -1\right]$$

$$\text{Out}[\cdot] = \frac{1}{(1998 - 79\text{i}) + (1186 + 99\text{i})J}$$

hence, we have verified that determinant of the inverse matrix is equal to inverse of determinant of the matrix that is $\det(X^{-1}) = 1/\det(X)$.

2.11

For the bicomplex numbers A, B, H and P defined in 2.10 above, show that the determinant of a BC-matrix $\begin{pmatrix} A & B \\ H & P \end{pmatrix}$ is same as the sum of the determinants of idempotent matrices, OR, $\det(X) = \det(1_X)e_1 + \det(2_X)e_2$,

$$A = (2 + 3\text{i}) + J*(59 + \text{i})$$
$$B = (3 + 5\text{i}) + J*(23 + 2\text{i})$$
$$H = (2 + 9\text{i}) + J*(25 - \text{i})$$
$$P = (23 + 8\text{i}) + J*(-23J + 3\text{i})$$

Let us find idempotent parts of these bicomplex numbers A, B, H and P

$$\text{In}[\cdot] := 1_A = 2 + 3\text{i} - \text{i}(59 + \text{i})$$
$$\text{Out}[\cdot] = 1_{(2+3\text{i})+(59+\text{i})J} = 3 - 56\text{i}$$

Similarly, define $2_A, 1_B, 2_B, 1_H, 2_H, 1_P, 2_P$.

We want to find $\det\left[\begin{pmatrix} 1_A & 1_B \\ 1_H & 1_P \end{pmatrix}\right]$,

$$\text{In}[\cdot] := \det\left[\begin{pmatrix} 3 - 56\text{i} & 5 - 18\text{i} \\ 1 - 16\text{i} & 26 + 31\text{i} \end{pmatrix}\right]$$
$$\text{Out}[\cdot] = 2097 - 1265\text{i}$$

Here we want to find $\det\left[\begin{pmatrix} 2_A & 2_B \\ 2_H & 2_P \end{pmatrix}\right]$

$$\text{In}[\cdot] := \det\left[\begin{pmatrix} 1 + 62\text{i} & 1 + 28\text{i} \\ 3 + 34\text{i} & 20 - 15\text{i} \end{pmatrix}\right]$$
$$\text{Out}[\cdot] = 1899 + 1107\text{i}$$

Here we are computing $\det\left[\begin{pmatrix} 1_A & 1_B \\ 1_H & 1_P \end{pmatrix}\right].e1$

$$\text{In}[\cdot] := \text{Simplify}\ [(2097 - 1265\text{i}) * (1 + \text{i} * J)/2]$$
$$\text{Out}[\cdot] = \left(\frac{1265}{2} + \frac{2097\text{i}}{2}\right)(-\text{i} + J)$$

(*Here we are computing $\det\left[\begin{pmatrix} 2_A & 2_B \\ 2_H & 2_P \end{pmatrix}\right].E2^*$)

$$\text{In}[\cdot] := \text{Simplify}\ [(1899 + 1107\text{i}) * (1 - \text{i} * J)/2]$$
$$\text{Out}[\cdot] = \left(\frac{1107}{2} - \frac{1899\text{i}}{2}\right)(\text{i} + J)$$
$$\text{In}[\cdot] := \text{Simplify}\left[\begin{array}{l}\left(\frac{1265}{2} + \frac{2097\text{i}}{2}\right)(-\text{i} + J) \\ + \left(\frac{1107}{2} - \frac{1899\text{i}}{2}\right)(\text{i} + J)\end{array}\right]$$
$$\text{Out}[\cdot] = (1998 - 79\text{i}) + (1186 + 99\text{i})J$$

Hence we have verified that the determinant of a bicomplex matrix is same as the sum of the determinants of idempotent component matrices, OR, $\det(X) = \det(1_X)e_1 + \det(2_X)e_2$.

2.12 Eigen Values of a BC-Matrix

Let us consider the matrix defined in Section 2.11.

Example 2.12.1.

$$\text{In}[\cdot] := \text{Eigenvalues}\left[\begin{pmatrix} A & B \\ H & P \end{pmatrix}\right]$$
$$\text{Out}[\cdot] = \left\{\frac{1}{2}\left(\begin{array}{c}(25 + 11\text{i}) + (36 + 4\text{i})J \\ -\sqrt{2}\sqrt{\begin{array}{c}(130 + 179\text{i}) \\ -(1516 - 298\text{i})J \\ +(4514 - 110\text{i})J^2\end{array}}\end{array}\right), \frac{1}{2}\left(\begin{array}{c}(25 + 11\text{i}) + (36 + 4\text{i})J \\ +\sqrt{2}\sqrt{\begin{array}{c}(130 + 179\text{i}) - (1516 - 298\text{i})J \\ +(4514 - 110\text{i})J^2\end{array}}\end{array}\right)\right\}$$
$$\text{In}[\cdot] := \text{Simplify}\left[\frac{1}{2}\left(\begin{array}{c}(25 + 11\text{i}) + (36 + 4\text{i})J \\ -\sqrt{2}\sqrt{\begin{array}{c}(130 + 179\text{i}) \\ -(1516 - 298\text{i})J \\ +(4514 - 110\text{i})J^2\end{array}}\end{array}\right), j^2 = -1\right]$$
$$\text{Out}[\cdot] = \frac{1}{2}\left((25 + 11\text{i}) - \sqrt{\begin{array}{c}(-8768 \\ +578\text{i}) \\ -(3032 \\ -596\text{i})J\end{array}} + (36 + 4\text{i})J\right)$$
$$\text{In}[\cdot] := \text{Simplify}\left[\frac{1}{2}\left(\begin{array}{c}(25 + 11\text{i}) + (36 + 4\text{i})J \\ +\sqrt{2}\sqrt{\begin{array}{c}(130 + 179\text{i}) \\ -(1516 - 298\text{i})J \\ +(4514 - 110\text{i})J^2\end{array}}\end{array}\right), j^2 = -1\right]$$
$$\text{Out}[\cdot] = \frac{1}{2}\left((25 + 11\text{i}) + \sqrt{\begin{array}{c}(-8768 + 578\text{i}) \\ -(3032 - 596\text{i})J\end{array}} + (36 + 4\text{i})J\right)$$

therefore, eigen values of the BC-matrix $\begin{pmatrix} A & B \\ H & P \end{pmatrix}$ are

$$\left\{\frac{1}{2}\left((25 + 11\text{i}) - \sqrt{\begin{array}{c}(-8768 + 578\text{i}) \\ -(3032 - 596\text{i})J\end{array}} + (36 + 4\text{i})J\right), \frac{1}{2}\left((25 + 11\text{i}) + \sqrt{\begin{array}{c}(-8768 + 578\text{i}) \\ -(3032 - 596\text{i})\mathfrak{J}\end{array}} + (36 + 4\text{i})\mathfrak{J}\right)\right\}$$

Now let us find the product of eigen values of this matrix

$$\text{In}[\cdot] := \left(\frac{1}{2}\left((25 + 11\text{i}) - \sqrt{\begin{array}{c}(-8768 + 578\text{i}) \\ -(3032 - 596\text{i})J\end{array}} + (36 + 4\text{i})J\right)\right) * \left(\frac{1}{2}\left((25 + 11\text{i}) + \sqrt{\begin{array}{c}(-8768 + 578\text{i}) \\ -(3032 - 596\text{i})J\end{array}} + (36 + 4\text{i})J\right)\right)$$
$$\text{Out}[\cdot] = \frac{1}{4}\left((25 + 11\text{i}) - \sqrt{\begin{array}{c}(-8768 \\ +578\text{i}) \\ -\left(\begin{array}{c}3032 \\ -596\text{i}\end{array}\right)J\end{array}} + (36 + 4\text{i})J\right) \left((25 + 11\text{i}) + \sqrt{\begin{array}{c}(-8768 + 578\text{i}) \\ -(3032 - 596\text{i})J\end{array}} + (36 + 4\text{i})J\right)$$
$$\text{In}[\cdot] := \text{Simplify}\left[\frac{1}{4}\left((25 + 11\text{i}) - \sqrt{\begin{array}{c}(-8768 \\ +578\text{i}) \\ -(3032 \\ -596\text{i})J\end{array}} + (36 + 4\text{i})J\right)\left(\begin{array}{c}(25 + 11\text{i}) + \sqrt{\begin{array}{c}(-8768 + 578\text{i}) \\ -(3032 - 596\text{i})J\end{array}} \\ +(36 + 4\text{i})J\end{array}\right), j^2 = -1\right]$$
$$(1998 - 79\text{i}) + (1186 + 99\text{i})J$$

(* which is same as determinant of the matrix*)

hence, we have shown here that the product of eigen values is equal to the determinant of the given BC-matrix.
Now, let us find the eigen values of a given matrix by traditional method i.e., by finding the roots of the characteristic equation.

$$\text{In}[\cdot] := \det\left[\begin{pmatrix} A & B \\ H & P \end{pmatrix} - \mu * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$\text{Out}[\cdot] = (61+48\text{i}) + (1186+99\text{i})J - (1937-127\text{i})J^2 - (25+11\text{i})\mu - (36+4\text{i})J\mu + \mu^2$$

$$\text{In}[\cdot] := \text{Simplify}\,[(61+48\text{i}) + (1186+99\text{i})J, - (1937-127\text{i})J^2 - (25+11\text{i})\mu - (36+4\text{i})J\mu + \mu^2\; J^2 = -1]$$

$$(1998-79\text{i}) + j((1186+99\text{i}) - (36+4\text{i})\mu) - (25+11\text{i})\mu + \mu^2$$

(this is the characteristic polynomial)

$$\text{In}[\cdot] := \text{Solve}\left[(1998-79\text{i}) + j((1186+99\text{i}) - (36+4\text{i})\mu) - (25+11\text{i})\mu + \mu^2 = 0, \mu\right]$$

$$\left\{\left\{\mu \to \frac{1}{2}\left((25+11\text{i}) + (36+4\text{i})J - \sqrt{2}\sqrt{(-3744+433\text{i}) - (1516-298\text{i})J + (640+144\text{i})J^2}\right)\right\}\right.$$

$$\left.\left\{\mu \to \frac{1}{2}\left((25+11\text{i}) + (36+4\text{i})J + \sqrt{2}\sqrt{(-3744+433\text{i}) - (1516-298\text{i})J + (640+144\text{i})J^2}\right)\right\}\right\}$$

These are characteristic roots or characteristic values or eigen values. let us simplify them for J.

$$\text{In}[\cdot] := \text{Simplify}\left[\frac{1}{2}\left((25+11\text{i}) + (36+4\text{i})J - \sqrt{2}\sqrt{(-3744+433\text{i}) - (1516-298\text{i})J + (640+144\text{i})J^2}\right)\; j^2 = -1\right]$$

$$\text{Out}[\cdot] = \frac{1}{2}\left((25+11\text{i}) - \sqrt{(-8768+578\text{i}) - (3032-596\text{i})J} + (36+4\text{i})J\right)$$

$$\text{In}[\cdot] := \text{Simplify}\left[\frac{1}{2}\left((25+11\text{i}) + (36+4\text{i})J + \sqrt{2}\sqrt{(-3744+433\text{i}) - (1516-298\text{i})J + (640+144\text{i})J^2}\right)\; j^2 = -1\right]$$

$$\text{Out}[\cdot] = \frac{1}{2}\left((25+11\text{i}) + \sqrt{(-8768+578\text{i}) - (3032-596\text{i})J} + (36+4\text{i})J\right)$$

Observation. Eigen values found by both the methods are same.
Let us find the sum of eigen values

$$\text{In}[\cdot] := \frac{1}{2}\left((25+11\text{i}) - \sqrt{(-8768+578\text{i}) - (3032-596\text{i})J} + (36+4\text{i})J\right) + \frac{1}{2}\left((25+11\text{i}) + \sqrt{(-8768+578\text{i}) - (3032-596\text{i})J} + (36+4\text{i})J\right)$$

$$\text{Out}[\cdot] = \frac{1}{2}\left((25+11\text{i}) - \sqrt{(-8768+578\text{i}) - (3032-596\text{i})J} + (36+4\text{i})J\right) + \frac{1}{2}\left((25+11\text{i}) + \sqrt{(-8768+578\text{i}) - (3032-596\text{i})J} + (36+4\text{i})J\right)$$

$$\text{In}[\cdot] := \text{Simplify}\left[\frac{1}{2}\left((25+11\text{i}) - \sqrt{(-8768+578\text{i}) - (3032-596\text{i})J} + (36+4\text{i})J\right) + \frac{1}{2}\left((25+11\text{i}) + \sqrt{(-8768+578\text{i}) - (3032-596\text{i})J} + (36+4\text{i})J\right), j^2 = -1\right]$$

$$\text{Out}[\cdot] = (25 + 11\text{ if}) + (36+4\text{i})J$$

which is the sum of principal diagonal elements of the matrix (or the trace of the matrix) i.e., $A = (2+3\text{i}) + j*(59+\text{i})$ and $\text{P} = (23+8\text{i}) + j*(-23J+3\text{i})$.

Example 2.12.2. Let us consider the matrix $\begin{pmatrix} I*J & 0 \\ I & J \end{pmatrix}$ and find its eigen values by both the methods.

$$\text{In}[\cdot] := \det\left[\begin{pmatrix} I*J & 0 \\ I & J \end{pmatrix}\right]$$

$$\text{i}J^2\ (*= -i\,*)$$

$$\text{In}[\cdot] := \text{Eigenvalues}\left[\begin{pmatrix} \text{i}*J & 0 \\ \text{i} & J \end{pmatrix}\right]$$

$$\text{Out}[\cdot] = \{\text{i}J, J\}$$

$$\text{In}[\cdot] := \det\left[\begin{pmatrix} \text{i}*J & 0 \\ \text{i} & J \end{pmatrix} - \lambda * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$\text{Out}[\cdot] = \text{i}J^2 - (1+\text{i})J\lambda + \lambda^2$$

$$\text{In}[\cdot] := \text{Simplify}[\text{i})J^2 - (1+\text{i})J\lambda + \lambda^2, J^2 == -1]$$

$$\text{Out}[\cdot] = -\text{i} - (1+\text{i})J\lambda + \lambda^2$$

$$\text{In}[\cdot] := \text{Solve}[-\text{i} - (1+\text{i})j\lambda + \lambda^2 == 0, \lambda]$$

$$\text{Out}[\cdot] = \left\{\left\{\lambda \to \frac{1}{2}\left((1+\text{i})J - \sqrt{2}\sqrt{2\text{i}+\text{i}j^2}\right)\right\}, \left\{\lambda \to \frac{1}{2}\left((1+\text{i})J + \sqrt{2}\sqrt{2\text{i}+\text{i}j^2}\right)\right\}\right\}$$

$$\text{In}[\cdot] := \text{Simplify}\left[\frac{1}{2}\left((1+\text{i})J - \sqrt{2}\sqrt{2\text{i}+\text{i}J^2}\right), j^2 == -1\right]$$

$$\text{Out}[\cdot] = \left(\frac{1}{2} + \frac{\text{i}}{2}\right)(-1+j)$$

$$\text{In}[\cdot] := \text{Simplify}\left[\frac{1}{2}\left((1+\text{i})J + \sqrt{2}\sqrt{2\text{i}+\text{i}J^2}\right), J^2 == -1\right]$$

$$\text{Out}[\cdot] = \left(\frac{1}{2} + \frac{\text{i}}{2}\right)(1+J)$$

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Thus, the eigen values of the given matrix are $\left(\frac{1}{2}+\frac{i}{2}\right)(-1+J)$ and $\left(\frac{1}{2}+\frac{i}{2}\right)(1+J)$ which are different from the earlier values. So we can say that the process gives all the n^2 eigen values of the BC-matrix.

$\left(\frac{1}{2}+\frac{i}{2}\right)(-1+J) * \left(\frac{1}{2}+\frac{i}{2}\right)(1+J)$(* here we are finding the product of eigen valeus*)

Out [·] = $\frac{1}{2}i(-1+J)(1+J)$

In [·]:= Simplify $\left[\frac{1}{2}i(-1+J)(1+J), J^2 = -1\right]$

Out [·] = $-i$

It can be easily verified that the sum of eigen values is equal to the sum of the principal diagonal elements or the trace of the matrix.

Thus, the product of these eigen values is equal to the determinant of the matrix. Hence in both the cases the product of eigen values is equal to the determinant but the eigen values are not same by both methods. In our earlier example where the matrix was $\begin{pmatrix} A & H \\ B & P \end{pmatrix}$, the eigen values were same by both the methods.

III. Conclusion

We have given step-by-step commands to find the determinant and Eigen values of a $n \times n$ BC-matrix. The concept of unimodular BC-matrix has been given with example. We have given various examples of singular and nonsingular BC-matrices. Eigen values have been computed by two different methods, and it was found that in one case the eigen values are same by both the methods while in the other bematrix the values were different by different methods but the product of eigen values is same as the determinant in both the cases. The relationship between eigen values and trace of a BC-matrix has also been verified with the help of Mathematica. We can compute the determinant, inverse and eigen values of a higher order BC-matrix easily with the help of Mathematica. We have also observed that simplifying the determinant of a BC -matrix for J and then finding the J -derivative is not giving the same outcome as finding the J derivatives (twice) of the determinant.

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